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At the vertex, $x'=0$, $y'=0$, and (2) becomes

$$x^2+y^2-px=0 \qquad (3).$$

If $(x_1,\ y_1)$ be any point on the required locus, its polar with respect to (1) is

$$px-2y_1y+px_1=0 \qquad (4).$$

The condition that (4) touches (3) is

$$x_1{}^2-y_1{}^2-px_1=0 \qquad (5),$$

an equilateral hyperbola.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland; CHAS. C. PURYEAR, Professor of Mathematics, Agricultural and Mechanical College, College Station, Texas; and G. B. M. ZERR, Texarkana, Ark.

Let $y^2=4ax$ be the equation to the parabola, $(b,\ c)$ any point.

Then $cy=2a(x+b)$.....(1) is the polar of $(b,\ c)$. $x^2+y^2=ax$.......(2) is the circle of curvature at the vertex. The value of y from (1) in (2) gives

$$c^2x^2+4a^2x^2+8a^2bx+4a^2b^2=ac^2x \qquad (3).$$

From (3) we find the condition that (1) should be tangent to (2) to be

$$a^2c^4=8a^3bc^2+16a^2b^2c^2. \quad \therefore\ c^2=8ab+16b^2.$$

$$\therefore\ a^2=(4b+a)^2-c^2 \qquad (4).$$

(4) represents an equilateral hyperbola.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

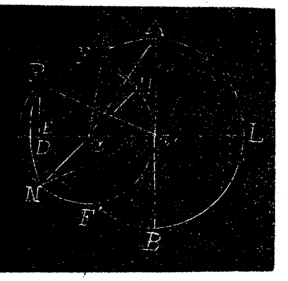
55. Proposed by GEORGE LILLEY, Ph. D., LL. D., Principal of Park School, 394 Hall Street, Portland, Oregon.

A horse is tethered by a rope, a feet long, fastened to a post in a circular fence enclosing a circular piece of grouhd b feet in diameter. If the horse is tethered outside of the fence over how much ground can he feed? If he is inside the fence over how much ground can he feed? b is greater than a in each case.

I. Solution by the PROPOSER.

Consider area $C'HFMA$. $C'A=C'P=a$, feet. $CC'=CH=\frac{1}{2}b$, feet. Let $\angle HCF=\phi$. $HM=$arc $HF=\frac{1}{2}b\phi$. The area element is $\frac{1}{2}(HM)^2d\phi$.

$$\therefore \text{ area } C'HFMA=\tfrac{1}{8}\int_0^{\frac{2a}{b}} b^2\phi^2 d\phi,=\frac{a^3}{3b}.$$



Hence, when the horse is outside, he can graze over

$$\frac{a^2}{6b}(4a+3b\pi), \text{ square feet.}$$

Draw PE at right angles to DC'. Let $x=CE$, and $\angle PC'E=\theta$.

$$\text{Area of sector } PC'D=\int_0^\theta\int_0^a da d\theta,=\tfrac{1}{2}a^2\cos^{-1}\frac{b+2x}{2a},=\text{sector } NC'D.$$

$$\text{Area of sector } PCC'HF=\tfrac{1}{8}b^2\times\text{angle } PCC',=\tfrac{1}{8}b^2\cos^{-1}\left(-\frac{2x}{b}\right).$$

$$\text{Area of triangle } PCC'=\tfrac{1}{8}b\sqrt{b^2-4x^2}.$$

$$\therefore \text{ area of segment } PFHC'=\tfrac{1}{8}b^2\cos^{-1}\left(-\frac{2x}{b}\right)-\tfrac{1}{8}b\sqrt{b^2-4x^2},=\text{segment } HC'F'.$$

$$x=\frac{2a^2-b^2}{2b}.$$

Hence, area of $C'HPNF'=a^2\cos^{-1}\frac{a}{b}+\frac{1}{4}b^2\cos^{-1}\frac{b^2-2a^2}{b^2}-\frac{1}{2}a\sqrt{b^2-a^2}$, square feet; the space grazed over inside the fence.

II. Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

In first part required area equals that of semicircle $ALB+2\times AMFHC$. Let HM be a portion of rope unwound and equal to HF. Let $\angle HCF=\phi$, $\angle MCF=\theta$, $CM=\rho$, and take CF for polar axis. Then

$$\rho^2=\tfrac{1}{4}b^2+\overline{HM}^2,=\tfrac{1}{4}b^2+\tfrac{1}{4}(b^2\phi^2).$$

$$\phi=\frac{2}{b}\sqrt{4\rho^2-b^2}. \qquad \frac{b}{2\rho}=\cos(\phi-\theta),=\cos\left(\frac{2}{b}\sqrt{4\rho^2-b^2}-\theta\right).$$

$$\theta=\frac{2}{b}\sqrt{4\rho^2-b^2}-\cos^{-1}\left(\frac{b}{2\rho}\right).$$

$$\frac{d\theta}{d\rho}=\frac{4\rho}{b\sqrt{4\rho^2-b^2}}-\frac{b}{2\rho^2\sqrt{1-\dfrac{b^2}{4\rho^2}}},\ =\frac{\sqrt{4\rho^2-b^2}}{b\rho}.$$

$$\frac{dA}{d\rho}=\frac{dA}{d\theta}\,\frac{d\theta}{d\rho},\ =\frac{\rho^2}{2}\times\frac{\sqrt{4\rho^2-b^2}}{b\rho},\ =\frac{\rho\sqrt{4\rho^2-b^2}}{2b}.$$

Limits of ρ for $CFMA$ are seen to be $\frac{1}{2}b$ and $\sqrt{a^2+\frac{1}{4}b^2}$.

$\therefore$ Area $CFMA=\dfrac{1}{2b}\displaystyle\int_{\frac{1}{2}b}^{\sqrt{a^2+\frac{1}{4}b^2}}\rho\sqrt{4\rho^2-b^2}\,d\rho,\ =\dfrac{a^3}{3b}.$

Area $FHCAM=CFMA+C'CA-CFHC',=a^3/3b+ab/4-ab/4=a^3/3b.$

$\therefore$ Area $F'BLAFHC'=2a^3/3b+\pi a^2/2.$

Internal area is composed of $2\times$segment $PHC'+$sector $PDNC'$

$$\sin\tfrac{1}{2}\angle PCC'=(\tfrac{1}{2}a/\tfrac{1}{2}b)=a/b.\quad \angle PCC'=2\sin^{-1}(a/b).$$

$$\angle PCN=2\pi-4\sin^{-1}(a/b).\quad \angle PC'N=\pi-2\sin^{-1}(a/b).$$

Sec. $PDNC'=\dfrac{a^2}{2}\left(\pi-2\sin^{-1}\dfrac{a}{b}\right)$, sector $CPHC'=\dfrac{b^2}{8}\left(2\sin^{-1}\dfrac{a}{b}\right),=\dfrac{b^2}{4}\sin^{-1}\left(\dfrac{a}{b}\right)$

$$\triangle PCC'=\tfrac{1}{2}a\sqrt{\tfrac{1}{4}b^2-\tfrac{1}{4}a^2},=\tfrac{1}{4}a\sqrt{b^2-a^2}.$$

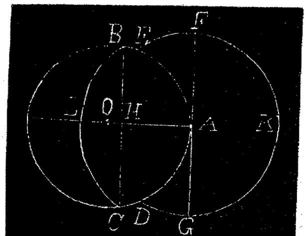
$$\text{Segment }PHC'=\frac{b^2}{4}\sin^{-1}\frac{a}{b}-\tfrac{1}{4}a\sqrt{b^2-a^2}.$$

$\therefore$ Internal area $=\dfrac{a^2}{2}\left(\pi-2\sin^{-1}\dfrac{a}{b}\right)+\dfrac{b^2}{2}\sin^{-1}\dfrac{a}{b}-\tfrac{1}{2}a\sqrt{b^2-a^2},$

$$=\tfrac{1}{2}\pi a^2-\frac{a}{2}\sqrt{b^2-a^2}+(\tfrac{1}{2}b^2-a^2)\sin^{-1}\left(\frac{a}{b}\right).$$

III. Solution by G. B. M. YERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let A be the point where the horse is tethered. $AF=a$, $AO=b/2$. Area $EADGKFE=2$ area $EAF+$area of semicircle GKF.



$\therefore A=\displaystyle\int\rho^2d\theta+\tfrac{1}{2}\pi a^2$; but $\rho=\frac{1}{2}b\theta$.

$\therefore\ A=\frac{1}{4}b^2\displaystyle\int_0^{2a/b}\theta^2d\theta+\tfrac{1}{2}\pi a^2,=\dfrac{a^2}{6b}(4a+3\pi b).$

Let $x^2+y^2=\frac{1}{4}b^2$, be the equation to circle center O. $(x-\frac{1}{2}b)^2+y^2=a^2$, be the equation to circle center A.

$$\therefore\ OH=\frac{b^2-2a^2}{2b},\quad \therefore\ BH=\frac{a}{b}\sqrt{b^2-a^2}.$$

A'=area of segment BLC+area of segment BAC,

$$=\frac{b^2}{4}\left\{\sin^{-1}\left(\frac{2a}{b^2}\sqrt{b^2-a^2}\right)-\frac{2a(b^2-2a^2)\sqrt{b^2-a^2}}{b^4}\right\}$$

$$+a^2\left\{\sin^{-1}\frac{\sqrt{b^2-a^2}}{b}-\frac{a\sqrt{b^2-a^2}}{b^2}\right\},$$

$$=\tfrac{1}{4}b^2\cos^{-1}\left(\frac{b^2-2a^2}{b^2}\right)+a^2\cos^{-1}\frac{a}{b}-\frac{a}{2}\sqrt{b^2-a^2}.$$

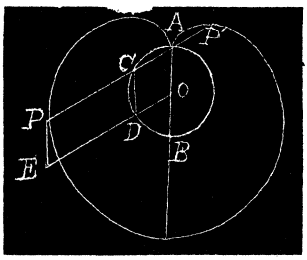
Also solved by *J. SCHEFFER* and *A. H. HOLMES.*

56. Proposed by B. F. BURLESON, Oneida Castle, New York.

Find (1) the length s of the closed curve of the cardioid; (2) its area A; (3) if made to revolve about its axis $2a$, find the maximum longitudinal circumference C of the solid generated; (4) find the surface K of the same; (5) its volume V; (6) the distance x_0 of the center of gravity of the solid from the origin O; and (7) the distance g_0 of the center of gravity of the plane curve from the origin O.

I. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland, and the PROPOSER.

Let $AB=a$ be the diameter of a circle. From A draw any chord AC. Make CP and $CP'=b$, then will the locus of P or P' be the Limaçon. If $AP=r$, $\angle PAB=\theta$, we find at once the polar equation of the Limaçon to be $r=a\cos\theta+b$. If $b>a$, the curve consists of but one loop; if $b<a$, it has two loops, and if $b=a$ the curve becomes the Cardioid, the polar equation of which is $r=a(1+\cos\theta)$. It can easily be shown that the cardioid is an epicycloid, the generating circle of which is equal to the fixed one; also, drawing through the center O of the circle a line parallel to AP cutting the circumference of the circle at D, and drawing through P a line parallel to CD, this line is a tangent to the cardioid at P.



The different problems proposed are best solved by means of the polar equation of the curve.

(1). The length $s=2\int_0^\pi d\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2},=2a\int_0^\pi \cos\tfrac{1}{2}\theta d\theta,=8a.$

(2). The area $A=\int_0^\pi r^2 d\theta,=a^2\int_0^\pi (1+\cos\theta)^2 d\theta,=\tfrac{3}{2}\pi a^2.$